A Deterministic Algorithm for Solving Imprecise Decision Problems

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Motivation

Problems with classical decision analysis tools:

- Numerically precise data is required.
- Data must be consistent.
Modelling Impreciseness

- **Dempster-Shafer theory** [Shafer, 76]:
  - Focus on representation—not decision analysis.
  - Unnecessarily strong with respect to interval representation.

- **Fuzzy set theory**, e.g. [Lai & Hwang, 94].

- **Epistemic reliability** [Gärdenfors & Sahlin, 83]:
  - Only imprecise probabilities—not utilities.
  - Only global beliefs.
Outline

- Representation
- Evaluation
- Dealing with Inconsistency
- Interval and Point Values
- Conclusions
A decision situation $D$ consists of a set of $n$ alternatives

$$\{\{c_{ij}\}_{j=1,...,m_i}\}_{i=1,...,n}$$

An alternative is represented by a set $C_i$ of $m_i$ consequences.

Each consequence $c_{ij}$ has a probability $p_{ij}$ of occurring, and has utility $u_{ij}$ for the decision maker if it occurs.
 Statements

▶ Exact statements:

“The probability of \( c_{21} \) is 0.17.” (\( p_{21} = 0.17 \))

▶ Qualitative statements:

“Consequence \( c_{11} \) is very probable.” (\( p_{11} \in [a, b] \))

▶ Comparative statements:

“Consequence \( c_{12} \) is at least as desirable as \( c_{11} \).”
(\( u_{12} \geq u_{11} \))
The set of constraints involving probability variables, together with $\sum_{j=1}^{m_i} p_{ij} = 1$ for each set of consequences, is the probability base $\mathcal{P}$.

The utility base $\mathcal{V}$ consists of all constraints involving utility variables.
Epistemologically Possible Distributions

- The solution set $E_P$ of $P$ is the set of epistemologically possible probability distributions.

- The solution set $E_V$ of $V$ is the set of epistemologically possible utility distributions.

These are the distributions in which the decision maker has a positive belief.

No variation in belief intensity!
Unit Cubes

- A unit cube for a consequence $c_{ij}$ is the interval $[0, 1]$, denoted $B = (b_{ij})$.

- A unit cube for an alternative $\{c_{ij}\}_{j=1,...,m_i}$ is the space $[0, 1]^{m_i}$, denoted $B = (b_{i1}, \ldots, b_{im_i})$.

- A unit cube for a decision situation $D$, given unit cubes $B_i$ for the alternatives, is the space $B_1 \times \ldots \times B_n$, denoted $B = (B_1, \ldots, B_n)$ (alt. $B = (b_1, \ldots, b_k)$).
Global Belief Distributions

Given a unit cube $B = (b_1, \ldots, b_k)$, a global belief distribution over $B$ is a positive distribution $g$ such that

$$\int_B g(x) dV_B(x) = 1$$

$$g(u_{11}, u_{12}) = \begin{cases} 
3(u_{11}^2 + u_{12}^2) & \text{if } u_{12} \geq u_{11} \\
0 & \text{otherwise}
\end{cases}$$
Global belief distributions generalizes the concept of probability and utility bases.

Given a global belief distribution \( g \), \( \text{supp } g \) is the epistemologically possible distributions (\( E_P \) or \( E_V \)).
Seldom, a decision maker can specify global belief distributions.

Given a unit cube \( B = (b_1, \ldots, b_k) \), a local belief distribution over a subspace \( b_i \) of \( B \) is a positive distribution \( f_i \) such that

\[
\int_{b_i} f_i(x_i) dV_{b_i}(x_i) = 1
\]
Define a local belief distribution over each probability variable $p_{ij}$ and each utility variable $u_{ij}$.

Specify relationships between variables using linear constraints of the form

$$
\sum_{i} a_i x_i \mathcal{R} b,
$$

where $\mathcal{R}$ is any of the relations $\equiv$, $\leq$, or $\geq$. 
Expected Utility

Given a decision situation $D$, the expected utility of an alternative $C_i$ is

$$E(C_i) = \sum_{j=1}^{m_i} p_{ij} u_{ij}$$

A decision maker adhering to the principle of maximizing expected utility chooses the alternative with the highest expected utility.
The centroid of a global belief distribution $g$ defined over a unit cube $B = (b_1, \ldots, b_k)$ is the vector $x_g = (\beta_1, \ldots, \beta_k)$ in $B$ whose $i$:th component is

$$\beta_i = \int_B x_i \cdot g(x) dV_B(x)$$

The centroid of a local belief distribution $f$ defined over the interval $b_i$ is

$$x_f = \int_{b_i} x_i \cdot f(x_i) dV_{b_i}(x_i)$$

Intuitively, the centroid is where the belief mass is concentrated.
The generalized expected mean value of an alternative $C_i$, given centroids $x_{p_i}$ for the belief distribution over $(p_{i1}, \ldots, p_{imi})$ and $x_{u_i}$ for the belief distribution over $(u_{i1}, \ldots, u_{imi})$, is

$$G(C_i) = \langle x_{p_i}, x_{u_i} \rangle = \sum_{j=1}^{m_i} x_{p_i}(j)x_{u_i}(j)$$
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Inconsistency

Given the vector $x_p$ consisting of the centroids $x_{p_{ij}}$ of the belief distributions over the probability variables, and a set $\mathcal{P}$ of linear constraints, it may be the case that the centroids fall outside $E_{\mathcal{P}}$ (e.g. $\sum_{j=1}^{m_i} x_{p_{ij}} \neq 1$).

When this happens, we can choose a vector $x'_p$ in $E_{\mathcal{P}}$ and use this when computing the generalized mean value.

What is a good choice for $x'_p$?
Measuring Inconsistency

Define an inconsistency measure $M(x)$, and let $x'_p$ be

$$x'_p = \text{argmin}_{x \in E_p} M(x)$$
Preferring Vectors Close to the Centroid

The following inconsistency measure expresses a bias towards vectors in $E_P$ that are close to $x_p$:

$$M(x) = \frac{1}{2} \|x - x_p\|^2.$$

(Constrained Least-Squares Problem)

**Rationale:** Since $x_p$ represents the center of belief mass, minimizing the Euclidean distance is likely to give the vector in $E_P$ which the decision maker has the highest belief in.
Deficiencies

- Does not handle inconsistent constraints (i.e. when $E_P = \emptyset$).

- Treats all constraints as **hard**. Only $\sum_{j=1}^{m_i} p_{ij} = 1$ are forced upon the decision maker by the axioms of probability. All other constraints are, just as the belief distributions, expressions of the decision maker’s beliefs.
Introduce soft constraints that are allowed to be relaxed by adding positive slack variables $\xi_i$:

- $\sum_i a_i x_i \leq b$ becomes $\sum_i a_i x_i \leq b + \xi_i$.
- $\sum_i a_i x_i \geq b$ becomes $\sum_i a_i x_i \geq b - \xi_i$.
- $\sum_i a_i x_i = b$ becomes $\sum_i a_i x_i \leq b + \xi_i$ and $\sum_i a_i x_i \geq b - \xi_{i+1}$.
A modified inconsistency measure dealing with soft constraints:

$$\tilde{M}(x, \xi_p) = \frac{1}{2} \|x - x_p\|^2 + C \sum_{i=1}^{\ell_p} \xi_p(i)$$

(Convex Quadratic Programming Problem)

The parameter $C$ allows the decision maker to control the penalty for modifying constraints, a larger $C$ expressing a preference for moving the centroid instead of relaxing constraints.
The vector $x'_p - x_p$ (and $\xi_p$ when applicable) can guide the decision maker when trying to reduce the inconsistency in the model.

The decision maker could be allowed to specify whether any constraints other than $\sum_{j=1}^{m_i} p_{ij} = 1$ should be treated as hard.
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Interval Values

▷ Assume symmetric belief distributions over intervals.

▷ Reduces complexity because coordinates of the centroids simply become the midpoints of the intervals.
Point Values

▷ We only have the constraints $\sum_{j=1}^{m_i} p_{ij} = 1$.

▷ Interpret point values as being the only points with positive belief.

▷ Minimizing the Euclidean distance to $x_p$ in accordance with $M(x)$ gives us

$$x'_{p_i(j)} = x_{p_i(j)} + \frac{1 - \sum_{j=1}^{m_i} x_{p_i(j)}}{m_i}$$
Conclusions

- Belief distributions—a versatile representation of impreciseness with simple semantics.

- Inconsistent decision models can be evaluated, and the severeness of the inconsistency is given by an inconsistency measure.

- The inconsistency measure can be changed to express different biases.

- Same principles can be applied to decision models with interval and point values.
Future Research

- Develop a better intuition for how belief distributions can be used.

- Learning and updating belief distributions given new evidence.

- Algorithms for sensitivity analysis.

- Alternative inconsistency measures (e.g. expressing a bias towards vectors with a high belief).